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On Jacobi's Method of Facilitating the Numerical Solution of Equations arising in the Theory of Secular Perturbations.

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1. The equations to be considered here are of the type

$$\Delta = 0$$

where Δ is a symmetric determinant which may be written in the form

$$\begin{vmatrix} A_{11} - \lambda, & A_{12}, & \dots & A_{1n} \\ A_{21}, & A_{22} - \lambda, & \dots & A_{2n} \\ - & - & \dots & - \\ A_{n1}, & A_{n2}, & \dots & A_{nn} - \lambda \end{vmatrix}$$

and which is no other than the familiar discriminating determinant which occurs in so many branches of mathematics. As it arises in the determination of the mean motions of the perihelion and the node, the number of rows and columns corresponds to the number of the planets. In the application to the solar system the determinant is of the eighth order; and little reflexion is necessary to see that its direct reduction would entail enormous labour. Hence Jacobi's method * of approximating to the numerical values of the roots without reducing the determinant is of great practical interest.

2. The principle of Jacobi's method may be briefly recalled. The vanishing of Δ is the condition of the consistency of the system of n linear equations

$$(A_{11} - \lambda)x_1 + A_{12}x_2 + \dots + A_{1n}x_n = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$A_{n1}x_1 + A_{n2}x_2 + \dots + (A_{nn} - \lambda)x_n = 0$$

where $A_{rs} = A_{sr}$. Let $[i]$ and $[k]$ denote the ith and kth equations. The substitution

$$x_i = \chi_i \cos a - \chi_k \sin a$$
$$x_k = \chi_i \sin a + \chi_k \cos a$$

is made and $[i]$ and $[k]$ are replaced by $[i] \cos a + [k] \sin a$ and $-[i] \sin a + [k] \cos a$. Then if

$$\cot 2a = (A_{kk} - A_{ii})/2A_{ik}$$

the eliminant of the equations thus transformed is a symmetric determinant of exactly the same *form* as before. If A'_{rs} be the constituent corresponding to A_{rs} in the transformed determinant, we have

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* 'Über ein leichtes Verfahren die in der Theorie der Säcularstörungen vorkommenden Gleichungen numerisch aufzulösen,' *Werke*, vol. vii. p. 111; Crelle, vol. xxx. (1846), p. 64.

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$$A'_{ik} = A'_{ki} = 0$$
$$A'^{2}_{ii} + A'^{2}_{kk} = A_{ii}^2 + A_{kk}^2 + 2A_{ik}^2$$
$$A'^{2}_{ij} + A'^{2}_{kj} = A_{ij}^2 + A_{kj}^2,\ j \neq i \neq k.$$

The result of the transformation is therefore to annihilate two equal constituents and to leave unaltered the sum of the squares of the other constituents outside the leading diagonal. The process can be repeated until the sum of the squares of the constituents becomes very small except in the leading diagonal, the constituents in which approximate to the factors of the determinant, and will accordingly give the roots of the equation. The approximations to the roots after a certain number of transformations can be completed by a different process, also due to Jacobi.

3. It may be interesting to study the theory underlying this method in a more general manner. For this purpose let the determinant Δ be multiplied by

$$\delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \text{—} & \text{—} & \dots & \text{—} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

the mode of composition being by rows of each determinant in such a way that rows of Δ appear in the corresponding rows of $\delta\Delta$. The constituent in the jth row and sth column of the product

$$= \sum_{i=1}^{i=n} A_{ji}\, a_{si} - \lambda a_{sj}$$

Next let the determinant $\delta\Delta$ be multiplied by δ, the mode of composition being by columns of $\delta\Delta$ and rows of δ. The constituent in the r^{th} row and s^{th} column of the final product

$$= \sum_{j=1}^{j=n} a_{rj} \left[\sum_{i=1}^{i=n} A_{ji}\ a_{si} - \lambda.\ a_{sj} \right]$$
$$= \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} A_{ji}\, a_{si}\, a_{rj} - \lambda. \sum_{j=1}^{j=n} a_{rj}\, a_{sj}.$$

This expression is unaltered by the interchange of r and s, and therefore the resulting determinant is symmetrical.

4. The determinant $\delta^2\Delta$ is not only symmetrical but is also of precisely the same form as Δ, provided that

$$\sum_{j=1}^{j=n} a_{rj}^2 = 1,\ (r = 1, 2, \dots, n)$$
$$\sum_{j=1}^{j=n} a_{rj}\, a_{sj} = 0,\ (r, s = 1, 2, \dots, n\ ;\ r \neq s)$$

That is, provided that δ is the modulus of an orthogonal transformation scheme

$$x_1 = a_{11}\, y_1 + a_{21}\, y_2 + \dots + a_{n1}\, y_n$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$x_n = a_{1n}\, y_1 + a_{2n}\, y_2 + \dots + a_{nn}\, y_n$$

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which is such that

$$x_1^2+x_2^2+\dots+x_n^2 \equiv y_1^2+y_2^2+\dots+y_n^2$$

If these conditions are fulfilled, with their well-known consequences, and if

$$\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} A_{ji}\, a_{si}\, a_{rj} = B_{rs}$$

then $\delta^2\Delta =$

$$\begin{vmatrix} B_{11}-\lambda, & B_{12}, & \dots & B_{1n} \\ B_{21}, & B_{22}-\lambda, & \dots & B_{2n} \\ - & - & \dots & - \\ B_{n1}, & B_{n2}, & \dots & B_{nn}-\lambda \end{vmatrix}$$

is identically equal to Δ, for $\delta^2 = 1$.

5. A number of identities may be deduced from this result. The coefficient of $(-\lambda)^{n-1}$ in Δ is

$$A_{11}+A_{22}+\ .\ .\ .\ +A_{nn}=\Sigma A_{rr}$$

Hence

$$\Sigma A_{rr}=\Sigma B_{rr}$$

Again the coefficient of $(-\lambda)^{n-2}$

$$=\Sigma(A_{rr}A_{ss}-A_{rs}^2)$$

Hence

$$[\Sigma A_{rr}]^2-2\Sigma(A_{rr}A_{ss}-A_{rs}^2)$$
$$=[\Sigma B_{rr}]^2-2\Sigma(B_{rr}B_{ss}-B_{rs}^2)$$

which gives

$$\Sigma A_{rr}^2+2\Sigma A_{rs}^2=\Sigma B_{rr}^2+2\Sigma B_{rs}^2$$

That is to say, the sum of the squares of the constituents is the same in both determinants. Moreover, the sum of the squares of the constituents in the rth row or column of $\delta^2\Delta$ is

$$\begin{aligned}\sum_s B_{rs}^2 &= \sum_s\Big(\sum_j\sum_i A_{ji}a_{si}a_{rj}\Big)^2 \\ &= \sum_s\sum_j\sum_i\sum_k\sum_l A_{ji}A_{kl}a_{rj}a_{rk}a_{si}a_{sl} \\ &= \sum_i\sum_j\sum_k A_{ji}A_{ki}a_{rj}a_{rk}\end{aligned}$$

for $\sum_s a_{si}a_{sl} = 0$ unless $l = i$, in which case the sum is unity.

6. The conditions which the n^2 coefficients of an orthogonal transformation must satisfy are $\frac{1}{2}n(n+1)$ in number. They can be so chosen as to satisfy in addition the $\frac{1}{2}n(n-1)$ conditions.

$$B_{rs}=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} A_{ji}\, a_{si}a_{rj}=0,\ (r, s = 1, 2, \ .\ .\ ., n\ ;\ r \neq s)$$

Then all the constituents outside the leading diagonal of $\delta^2\Delta$ vanish, and the roots of the equation in λ become B_{11}, B_{22}, . . ., B_{nn}, where

$$B_{rr}=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} A_{ji}a_{ri}a_r$$

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But the complete solution of the equation $\Delta = 0$ is required in order to find the coefficients of the substitution. The additional relations given above show that

$$A_{11}x_1^2+A_{22}x_2^2+\dots+2A_{12}x_1x_2+\dots\equiv B_{11}y_1^2\dots+B_{nn}y_n^2$$

i.e. the substitution is one which transforms the quadratic form into a sum of square terms. Hence

$$\begin{aligned}&(A_{11}-\lambda)x_1^2+(A_{22}-\lambda)x_2^2+\dots+2A_{12}x_1x_2\\&\quad+\dots\equiv(B_{11}-\lambda)y_1^2+\dots+(B_{nn}-\lambda)y_n^2\end{aligned}$$

If now $\lambda = B_{rr}$, y_r disappears, and therefore

$$\begin{aligned}&x_1\left[(A_{11}-\lambda)\frac{\partial x_1}{\partial y_r}+A_{12}\frac{\partial x_2}{\partial y_r}+\dots+A_{1n}\frac{\partial x_n}{\partial y_r}\right]\\&+x_2\left[A_{21}\frac{\partial x_1}{\partial y_r}+(A_{22}-\lambda)\frac{\partial x_2}{\partial y_r}+\dots+A_{2n}\frac{\partial x_n}{\partial y_r}\right]\\&+\dots\equiv 0\end{aligned}$$

Hence

$$\begin{aligned}(A_{11}-\lambda)a_{r1}+A_{12}a_{r2}+\dots+A_{1n}a_{rn}&=0\\A_{21}a_{r1}+(A_{22}-\lambda(a_{r2}+\dots+A_{2n}a_{rn}&=0\\\dots\dots\dots\dots\dots\dots&\end{aligned}$$

These lead again to the equation $\Delta = 0$. The successive substitution of the roots of the latter equation in λ in the set of linear equations will yield the n sets of n coefficients.

7. It is thus seen that the complete transformation requires a knowledge of the roots of the very equation whose solution is to be effected. But it is possible to use a partial transformation instead of a complete one. The set of quantities $x_1, \dots, x_n$ can be divided into two sets $x_1, \dots, x_m$ and $x_{m+1}, \dots, x_n$ of which the former is transformed as before, while the latter remains unchanged. This means that the terms of $\sum_i\sum_j A_{ji}x_ix_j$ which involve the first set only are transformed into a sum of squares, those involving the second set only are unchanged, and the product terms involving one quantity from each set retain their form with changed coefficients. The transformation as a whole, as well as in part, remains orthogonal, but the coefficients of the substitution are such that $a_{si}=0$ ($s \neq i$; s or $i > m$) and $a_{ss}=1$ ($s > m$). Three cases must be considered separately, and these results are easily deduced from the formula of § 3 :

(1) If $r>m$, $s>m$, then

$$B_{rs} = A_{rs}$$

(2) If $r>m$, $s\not>m$, then

$$B_{rs} = \sum_{i=1}^{i=m} A_{ri}\,a_{si}$$

(3) If $r\not>m$, $s\not>m$, then

$$B_{rs} = \sum_{j=1}^{j=m}\ \sum_{i=1}^{i=m} A_{ji}a_{si}a_{rj}$$

8. It is now possible to choose the m^2 quantities $a_{si}(s \ngtr m, i \ngtr m)$ according to the method of § 6 so as to make the terms $B_{rs}(r \neq s)$ of § 7 (3) vanish. In this way all the constituents of the minor formed of the first m rows and columns of Δ outside the leading diagonal can be made to vanish. Since by interchange of rows and columns the constituents of the leading diagonal can be made to take any order, it is clear that any minor on the leading diagonal can be transformed in this way. If the minor is a determinant of the mth order, the solution of an equation of the mth degree is required for the transformation and $m(m-1)$ constituents can be made to vanish. To the transformed minor the results of §§ 3–6 apply. The complementary minor on the leading diagonal will remain unaltered. As regards the outstanding constituents it is easily deduced either from § 7 (2) or from the formula in § 5 that the sum of the squares of constituents lying in a line parallel to a row or column of the transformed minor remains unaltered. This is the generalisation of the fact on which the principle of Jacobi's method depends. The sum of the constant terms in the axis of the transformed minor is unchanged, while the sum of their squares is increased by the sum of the vanishing constituents. Outside this minor the sum of the squares of the constituents is left, line for line, unchanged. Hence repetition of the process must cause the axial terms to approximate to the factors of the determinant.

9. Jacobi's method amounts to the use of the case in which $m = 2$. The auxiliary angle α which is introduced in § 2 has of course a simple geometrical meaning. It is the angle between the coordinate axes and the axes of the conic

$$A_{ii}x^2 + 2A_{ik}xy + A_{kk}y^2 = 1$$

and its connexion with the substitution theory is obvious. In Jacobi's method it is always possible to remove the numerically greatest pair of constituents outside the diagonal at each transformation. Hence if S is the sum of the $n(n-1)$ squares after the first transformation the square of the greatest $> S/(n-2)(n+1)$, and each further transformation will reduce the sum of the squares faster than in the ratio

$$1 : 1 - \frac{2}{(n-2)(n+1)}$$

In the general case of m substitutions the $m(m-1)$ constituent to be removed cannot be chosen independently, and after a certain number of transformations it may be necessary to include terms previously reduced. Short of this event the ratio of decrease of the sum of the squares may be expected to be at least

$$1 : 1 - \frac{m(m-1)}{(n-m)(n+m-1)}$$

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Doubtless Jacobi's case is the simplest, but it would be theoretically possible to make $m = 4$, a case involving the solution of a biquadratic and removing twelve constituents. The case of $m = 3$, depending on a cubic and removing 6 constituents, would seem to be quite practicable and useful at least in the earlier transformations. In any case, when the factors of any axial minor are known, all the constituents of that minor outside the axis can be removed. The necessary substitutions might probably be facilitated by the use of an arithmometer. But after the work of Harzer, who employed Jacobi's method, it is unlikely that an occasion for employing such processes will recur.

10. The principle of Jacobi's method contains implicitly a proof that the roots of a discriminating determinant are real. The proof supposes the method repeated an infinite number of times. The idea of a limit may be avoided by the foregoing theory. For this it is only necessary to notice that if the extra-diagonal terms of a minor of any order be removed as before explained, it is possible to extend the process to a minor of an order higher by unity. That is to say, the process can be extended successively from a minor of the second order, for which it is always possible, to the complete determinant. For let the extra-diagonal constituents of Δ be zero except in the first row and column. Then

$$\Delta = \left[1 - \frac{A_{12}^2}{(A_{11}-\lambda)(A_{22}-\lambda)} - \frac{A_{13}^2}{(A_{11}-\lambda)(A_{33}-\lambda)} - \ldots\right]\Pi$$

where Π is the product of the diagonal terms. In Δ put $\lambda = A_{rr}$. Then

$$\Delta = -A_{1r}^2 (A_{22}-A_{rr})(A_{33}-A_{rr}) \ldots (A_{nn}-A_{rr})$$

Suppose that A_{22}, A_{33} . . . are in increasing order of magnitude. If now λ is put equal to $-\infty$, A_{22}, A_{33}, . . ., A_{nn}, $+\infty$ in turn, it is easily seen that the resulting values of Δ are of alternating sign, and therefore Δ has real roots situated between the values, assigned to λ. And if A_{22}, A_{33}, . . . are not all different, the roots are still real. For if, for example, $A_{22} = A_{33}$, the expanded form of Δ shows that A_{22} is a root, and the order of Δ can be depressed by writing $A_{22}-\lambda$ only once in the diagonal and $\sqrt{[A_{12}^2 + A_{13}^2]}$ in the corresponding places of the first row and column. Hence, generally, a determinant of the form considered has real roots, and its extra-diagonal constituents can all be removed by a real substitution. Thus by successive steps it is proved that the roots of any discriminating determinant are real. It is not difficult to see that a similar method could be applied to the more general case of Langrange's determinant.

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1902 *December* 6.

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Ephemeris for Physical Observations of

Greenwich Noon.	P.	L – O.	B.	Apparent Diameter.			*d.*	Q.	B′.
				Equat. 2*a*.	Defect.	Polar 2*b*.			
1903.	°	°	°	″	″	″	°	°	°
Mar. 28	335·811	203·012	+0·928	34·08	0·07	31·88	5·40	246·36	+0·99
30	335·752	203·463	0·956	34·17	0·08	31·96	5·67	246·36	1·02
April 1	335·695	203·910	0·984	34·26	0·09	32·05	5·94	246·37	1·05
3	335·640	204·354	1·012	34·36	0·10	32·14	6·20	246·37	1·08
5	335·587	204·795	1·039	34·46	0·11	32·24	6·46	246·38	1·11
7	335·535	205·230	1·065	34·57	0·12	32·35	6·72	246·39	1·14
9	335·485	205·663	1·091	34·68	0·13	32·45	6·97	246·40	1·17
11	335·436	206·091	1·116	34·80	0·14	32·56	7·22	246·40	1·19
13	335·388	206·514	1·141	34·92	0·15	32·67	7·46	246·41	1·22
15	335·343	206·934	1·165	35·04	0·16	32·79	7·70	246·42	1·24
17	335·300	207·347	1·188	35·17	0·17	32·91	7·94	246·42	1·27
19	335·258	207·757	1·211	35·30	0·18	33·03	8·17	246·43	1·29
21	335·218	208·162	1·233	35·44	0·19	33·16	8·39	246·43	1·32
23	335·180	208·560	1·254	35·58	0·20	33·30	8·61	246·44	1·34
25	335·144	208·954	1·275	35·73	0·21	33·44	8·82	246·44	1·36
27	335·109	209·341	1·295	35·89	0·22	33·58	9·03	246·45	1·38
29	335·075	209·724	1·315	36·05	0·23	33·73	9·24	246·45	1·41
May 1	335·043	210·098	1·335	36·22	0·24	33·88	9·43	246·46	1·43
3	335·012	210·468	1·355	36·39	0·25	34·04	9·62	246·46	1·45
5	334·982	210·830	1·375	36·56	0·26	34·20	9·80	246·47	1·47
7	334·956	211·186	1·395	36·74	0·28	34·37	9·98	246·47	1·49
9	334·931	211·535	1·415	36·92	0·29	34·54	10·15	246·48	1·51
11	334·907	211·877	1·435	37·11	0·30	34·72	10·31	246·48	1·53
13	334·883	212·211	1·454	37·30	0·31	34·90	10·46	246·49	1·55
15	334·859	212·538	1·474	37·49	0·32	35·08	10·61	246·49	1·57
17	334·837	212·858	1·493	37·69	0·33	35·27	10·75	246·50	1·59
19	334·817	213·169	1·512	37·90	0·34	35·46	10·88	246·51	1·62
21	334·798	213·472	1·531	38·11	0·35	35·66	11·00	246·52	1·64
23	334·780	213·767	1·550	38·33	0·36	35·86	11·12	246·53	1·66
25	334·764	214·052	1·569	38·55	0·37	36·06	11·22	246·55	1·68
27	334·749	214·330	1·588	38·77	0·38	36·27	11·32	246·56	1·70
29	334·735	214·597	1·606	39·00	0·39	36·48	11·41	246·58	1·71
31	334·722	214·855	1·624	39·23	0·39	36·70	11·49	246·60	1·73
June 2	334·710	215·103	1·641	39·47	0·40	36·92	11·55	246·62	1·75
4	334·698	215·343	+1·659	39·71	0·40	37·15	11·61	246·64	+1·77